Self Organized Criticality → OV with MFE focus

P.H. Diamond

CMTFO, CASS and Physics; UCSD

APTWG 2016

N.B.:

- SOC = Self Organized Criticality in a rigorous sense
- ‘SOC’ = Set of concepts surrounding and related to SOC, necessarily amorphous

Also:

- Focus on fundamental concepts and MFE applications – primarily micro-turbulence, transport
- No survey of 'SOC' applications, i.e.: earthquakes, markets, neuroscience, …
- For magnetic self-organization, see lecture by Susanna Cappello
- Time limit → must neglect many interesting works

Outline

- What is SOC and why should we care?
- A brief intellectual history of ‘SOC’ and SOC → where did the concept emerge from?
- Basic model paradigms: Piles, Avalanches, Hydrodynamic Models
- ‘SOC’ in MFE
- ‘SOC’ and kinetics → intro to CTRW and fractional kinetics
- FAQ re ‘SOC’ in MFE
- Comments and suggestions

What is SOC?

(cf: Jensen)

- (Constructive)

  Slowly driven, interaction dominated threshold system

  Classic example: sandpile

- (Phenomenological)

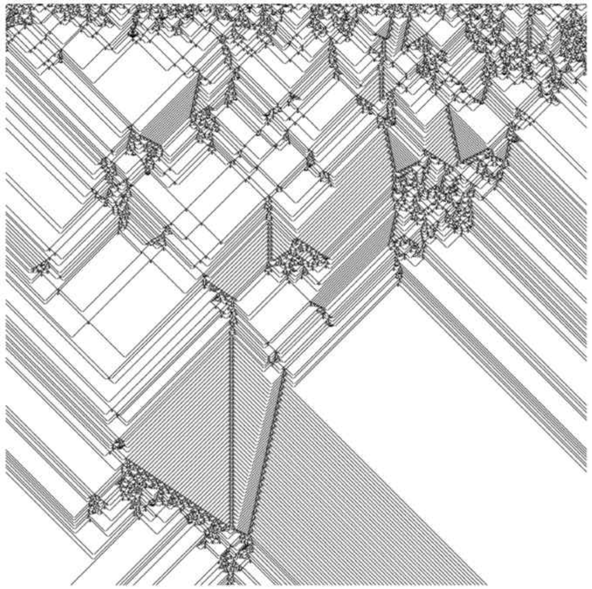
  System exhibiting power law scaling without tuning.

  Special note: 1/f noise; flicker shot noise of special interest

  See also: sandpile

N.B.: $1/f$ means $1/f^{\beta}$, $\beta \leq 1$

What is SOC?, cont'd



- Elements:

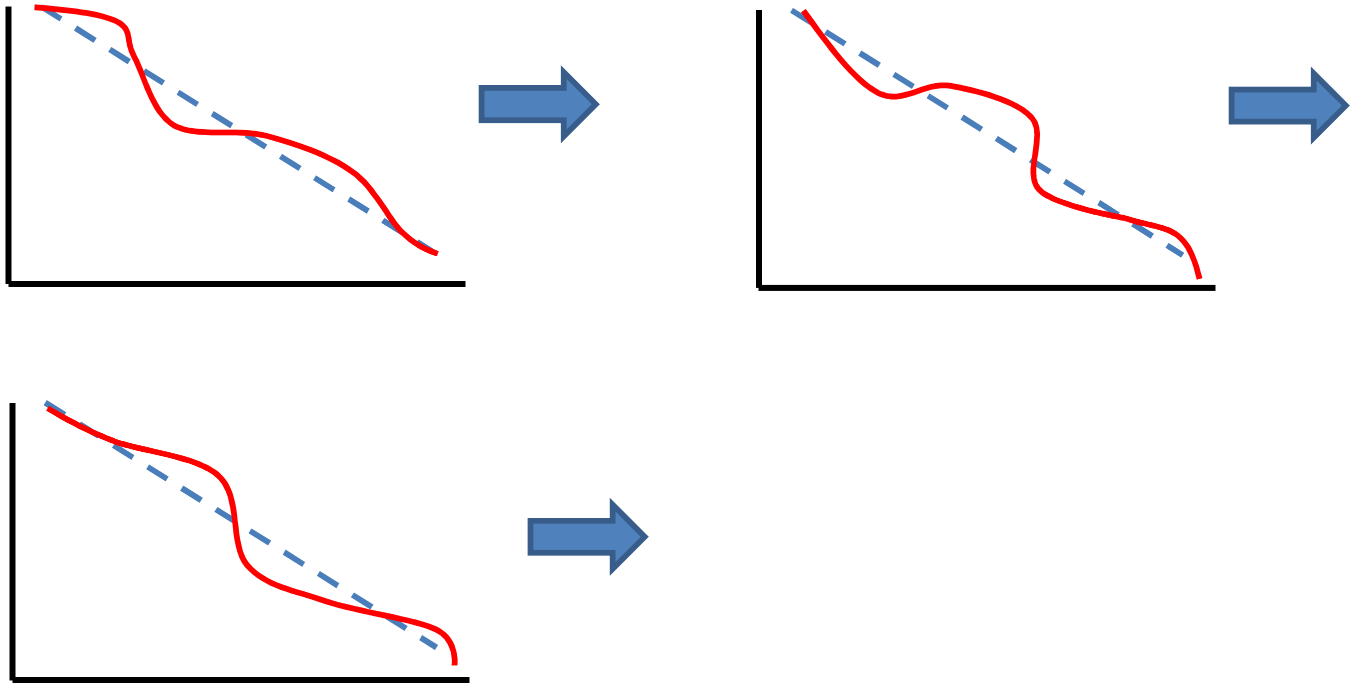
→ Interaction dominated

- – Many d-o-fs { Cells / Modes }
- – Dynamics dominated by d-o-f interaction i.e. couplings

→ Threshold and slow drive

- – Local criterion for excitation
- – Large number of accessible meta-stable, quasi-static configuration
- – 'Local rigidity' ←→ "stiffness" !?

- Multiple, metastable states

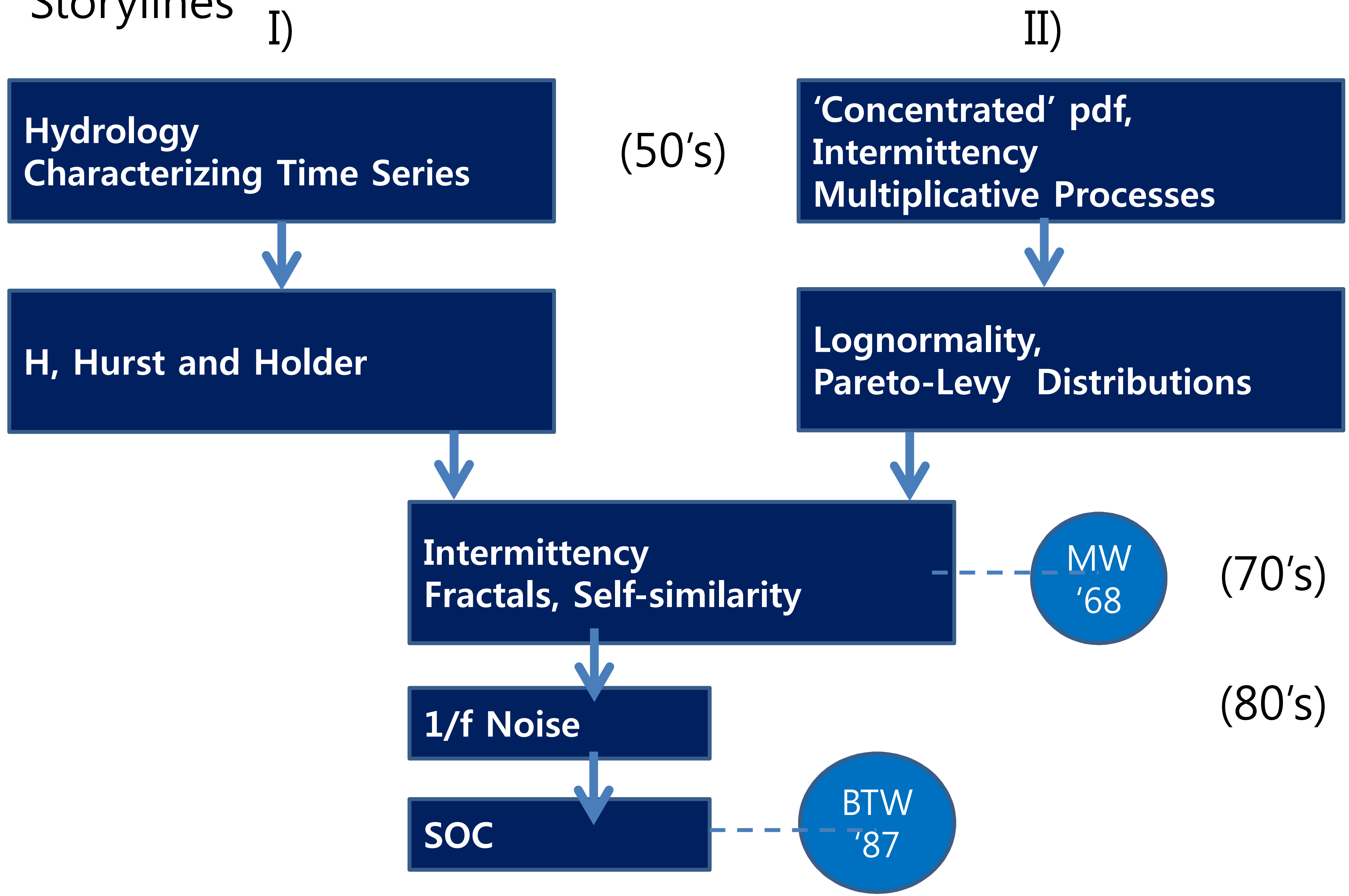


- Proximity to a 'SOC' state → local rigidity
- Unresolved: precise relation of 'SOC' state to marginal state

- Threshold and slow drive, cont'd
  - Slow drive uncovers threshold, metastability
  - Strong drive buries threshold – does not allow relaxation between metastable configurations
  - How strong is 'strong'? – set by toppling/mixing rules, box size, b.c. etc.
- Power law ←→ self-similarity
  - 'SOC' intimately related to:
    - Zipf's law: P(event) ~ 1/(size) (1949)
    - 1/f noise: S(f) ~ 1/f

A Brief Intellectual History of 'SOC'

- Storylines



I) Hydrology, Hurst, H

- Harold E. Hurst (1880-1978)
  - Hydrological engineer
  - Active in design, construction of Aswan High Dam; Egypt
- Concern:
  - Statistical description of Nile flow, discharge
  - Prediction, modelling based on time series → meticulous observation
- Problem:
  - Characterize time variation of reservoir content due river discharge

- Standard statistical distribution fails

  c.f. Mandelbrot, Wallis '68

Noah, Joseph, and Operational Hydrology

BENOIT B. MANDELBROT

JAMES R. WALLIS

International Business Machines Research Center
Yorktown Heights, New York 10598

Dedicated to Harold Edwin Hurst

. . . were all the fountains of the great deep broken up, and the windows of heaven were opened. And the rain was upon the earth forty days and forty nights. *Genesis, 6,* 11-12

. . . there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine . . . *Genesis, 41,* 29-30

Abstract. By 'Noah Effect' we designate the observation that extreme precipitation can be very extreme indeed, and by 'Joseph Effect' the finding that a long period of unusual (high or low) precipitation can be extremely long. Current models of statistical hydrology cannot account for either effect and must be superseded. As a replacement, 'self-similar' models appear very promising. They account particularly well for the remarkable empirical observations of Harold Edwin Hurst. The present paper introduces and summarizes a series of investigations on self-similar operational hydrology. (Key words: Statistics; synthesis; time series)

- Problem: "Noah", "Joseph" phenomena

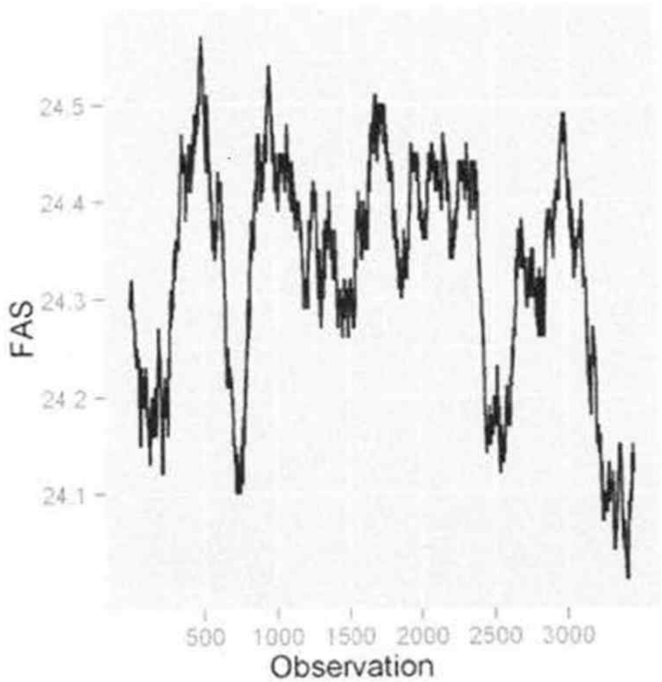
- Time series analysis

  $B_H(t)$ → general, stationary time series

  $E\{(B_H(t+T) - B_H(t))^2\} = T^{2H}$
- $H \equiv$ Hurst/Holder exponent
- Expected: H=1/2 → Brownian random walk
- Got: $0 < H < 1$, especially $1/2 < H < 1$ → Joseph, Noah effects

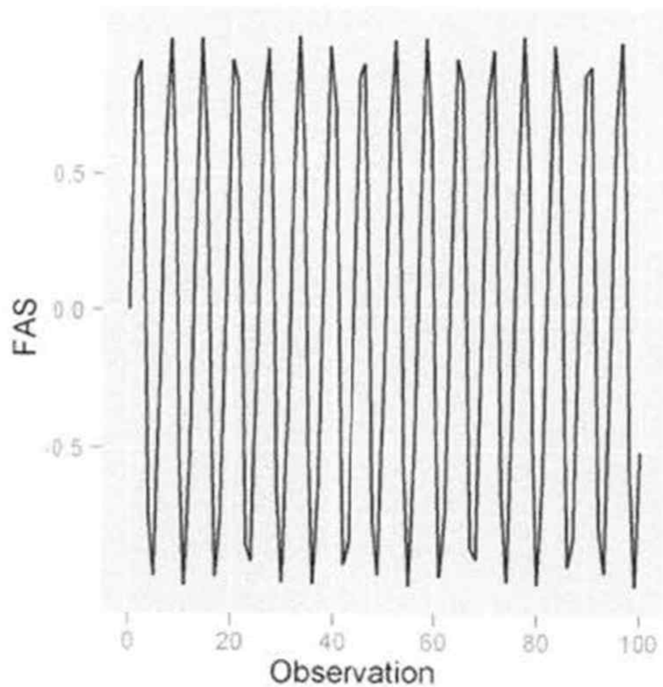
  → cyclic, non-periodic variability on all (time) series!
- $1/2 < H < 1$
  - Memory, positive correlation
  - Long term persistence
- $0 < H < 1/2$
  - Temporal anti-correlation
  - Hi/low value switching

- Some Examples



$H = 0.95 > 1/2$

→ Long term persistence



$H = 0.04 < 1/2$

→ Cycling, resembles LCO; anti-persistent

- Point: H measures memory in dynamics

- R/S Analysis
  - Time series $x_1, x_2, \ldots, x_n$
  - H defined by $cn^H = E\left\{\frac{R(n)}{S(n)}\right\}$, $n \equiv$ counts series elements
  - $R(n)$ = range of first n values = max($z_1, .., z_n$) – min($z_1, .., z_n$)
  - $z's$ = cumulative deviation from mean
  - $S(n)$ = standard deviation

Gini?

- Can define higher order Hurst coefficient, akin higher order structure functions/moments in turbulence
- Higher order moments reveal intermittency

- Further:
  - H related to fractal dimension of time series $1 < D < 2$,

    $D = 2 - H$
  - H related to frequency spectrum of the variation
  - $\langle (\Delta B)^2 \rangle_\omega \sim \omega^{-\alpha}$ $\alpha = 2H - 1$ $\begin{cases} H = 1/2 \rightarrow \text{white} \\ H = 1 \rightarrow 1/f\ ! \end{cases}$
- Enter 1/f issue!

  ↓

  Theme central to 'SOC'

II) Intermittency, Multiplicative Processes

- <u>Additive</u> processes → central limit theorem → gaussian statistics → Fokker-Planck theory etc. ➔ 'Mild' Randomness, and all is well, but boring...
- <u>Multiplicative</u> processes (i.e. avalanching) more interesting ➔ 'Wild' Randomness
- i.e. $x = \Pi_{i=1}^{N} x_i = x_1 x_2 \cdots x_N$ (c.f. Zeldovich et al)

  $x_i = 0 \; or \; 2$ each $P = 1/2$

- Then $\langle x \rangle = 1$

  $\langle x^2 \rangle = 2^N$

Point: $x = 0$ unless all $x_j = 2$
then $x = 2^N$ with $P = 2^{-N}$

➔ All non-zero probability concentrated in one outcome

- Welcome to intermittency! ➔ concentration of probability in limited set of events

- Intermittency includes "Noah', 'Joseph' phenomena…

- Multiplicative processes ←→ Log's additive

$$x = \Pi_{i=1}^{N} x_i$$

$$\log x = \log x_1 + \log x_2 + \cdots + \log x_N$$

Apply CLT to logs ➔ Lognormal distribution

$$F(\log x) = \exp\left[-\frac{\left(\log x - \overline{\log x}\right)^2}{2\sigma^2}\right] / (2\pi\sigma^2)^{1/2}$$

Assumes variance exists! → if not → Power law (Pareto-Levy)

- Lognormal ←→ Zipf ←→ 1/f related

i.e.

- $P\left(\frac{x}{\bar{x}}\right) = P(\log x)\frac{d\log x}{dx} = g\left(\frac{x}{\bar{x}}\right) d\left(\frac{x}{\bar{x}}\right)$

↑

Probability

$x/\bar{x}$ lies in $d(x/\bar{x})$ at $x/\bar{x}$

$\log(g) = -\log f$ + variance corrections

$$f = 1/(x/\bar{x})$$

- Lognormal well approximated by power law $P \sim \frac{1}{x}$ (Zipf's law), over finite range! (Montroll '82)
- Multiplicative processes related to Zipf's law trend
- Link to 1/f noise?

- 1/f Noise?

  A few observations:

  – Zipf and 1/f related but <u>different</u>

    Zipf → $P(\Delta B) \sim 1/|\Delta B|$

    $1/f$ → $\langle (\Delta B)^2 \rangle_\omega \sim 1\,/\,\omega$

    Both embody:

    - Self-similarity
    - Large events rare, small events frequent → intermittency phenomena
    - 1/f linked to H→1

  – 1/f noise (flickers, shot…)

    - Ubiquitous, suggests 'universality'
    - Poorly understood, circa 80's

- N.B.: Not easy to get 1/f …
- In usual approach to ω spectrum; ←→ (DIA, EDQNM, Dupree, Kadomtsev, Kraichnan, Krommes):

$$\langle \phi(t_1)\phi(t_2)\rangle = \left|\hat{\phi}\right|^2 e^{-|\tau|/\tau_c}$$

➔ $S(\omega) = \frac{1/\tau_c}{\omega^2 + 1/\tau_c^2} \sim \frac{1}{\omega^2}$

i.e. τ_c imposes scale, but 1/f scale free !?

- N.B.: Conserved order parameter may restore scale invariance
- But, consider ensemble of random processes each with own τ_c (Montroll, BTW)

$$S(\omega)_{eff} = \int_{\tau_{c1}}^{\tau_{c2}} P(\tau_c) S_{\tau_c}(\omega) d\omega$$

Probability of τ_c

- And... demand $P(\tau_c)$ scale invariant, i.e.

$$P(\tau_c) = d\tau_c/\tau_c$$

$$S(\omega) = \left.\frac{\tan^{-1}(\omega\tau_c)}{\omega}\right|_{\tau_{c1}}^{\tau_{c2}} \sim 1/\omega,$$ recovers 1/f !

→ but what does it mean? ...

- So, circa mid 80's, need a simple, intuitive model which:
  – Captures 'Noah', 'Joseph' effects in non-Brownian random process (H→1)
  – Display 1/f noise

SOC at last !

- Enter BTW '87:

Self-Organized Criticality: An Explanation of 1/*f* Noise (7000+ cites)

Per Bak, Chao Tang, and Kurt Wiesenfeld
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or 1/*f* noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

- Key elements:
  - Motivated by ubiquity and challenge of 1/f noise (scale invariant)
  - Spatially extended excitations (avalanches)

    Comment: statistical ensemble of collective excitations/avalanches is intrinsic
  - Evolve to 'self-organized critical structures of states which are barely stable'

    Comment: SOC state ≠ linearly marginal state!

    SOC state is <u>dynamic</u>

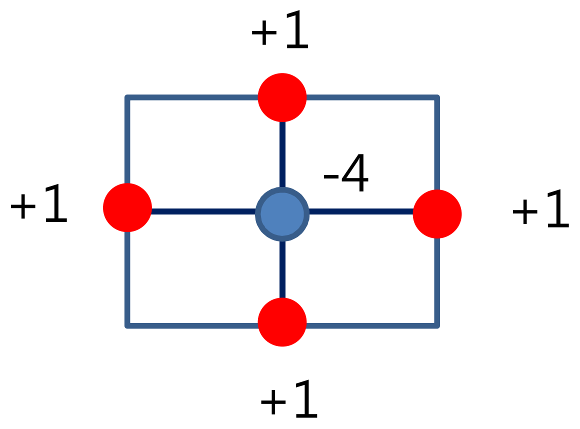
- Key elements, cont'd:
  - "The combination of <u>dynamical</u> minimal stability and spatial scaling leads to a power law for temporal fluctuations"
  - "Noise propagates through the scaling clusters by means of a "domino" effect upsetting the minimally stable states"

    Comment: space-time propagation of avalanching events
  - "The critical point in the dynamical systems studied here is an attractor reached by starting far from equilibrium: the scaling properties of the model"

    Comment: Noise essential to <u>probe</u> dynamic state

N.B.: BTW is example of well-written PRL

- Avalanches and Clusters:
  - BTW – 2D CA model



$Z \equiv \text{occupation}$

$Z > Z_{crt} = K$

$Z(x, y) \rightarrow Z(x, y) - 4$

$Z(x \pm 1, y) \rightarrow Z(x \pm 1, y) + 1$

$Z(x, y \pm 1) \rightarrow Z(x, y \pm 1) + 1$

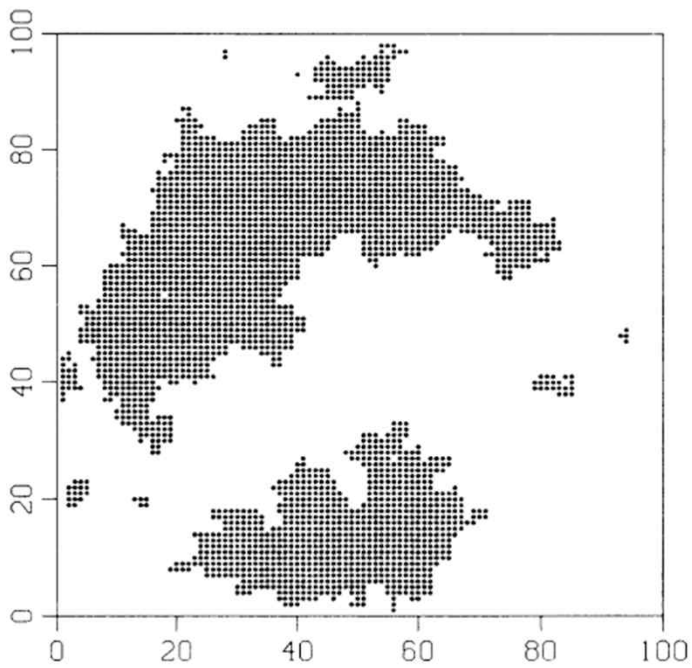
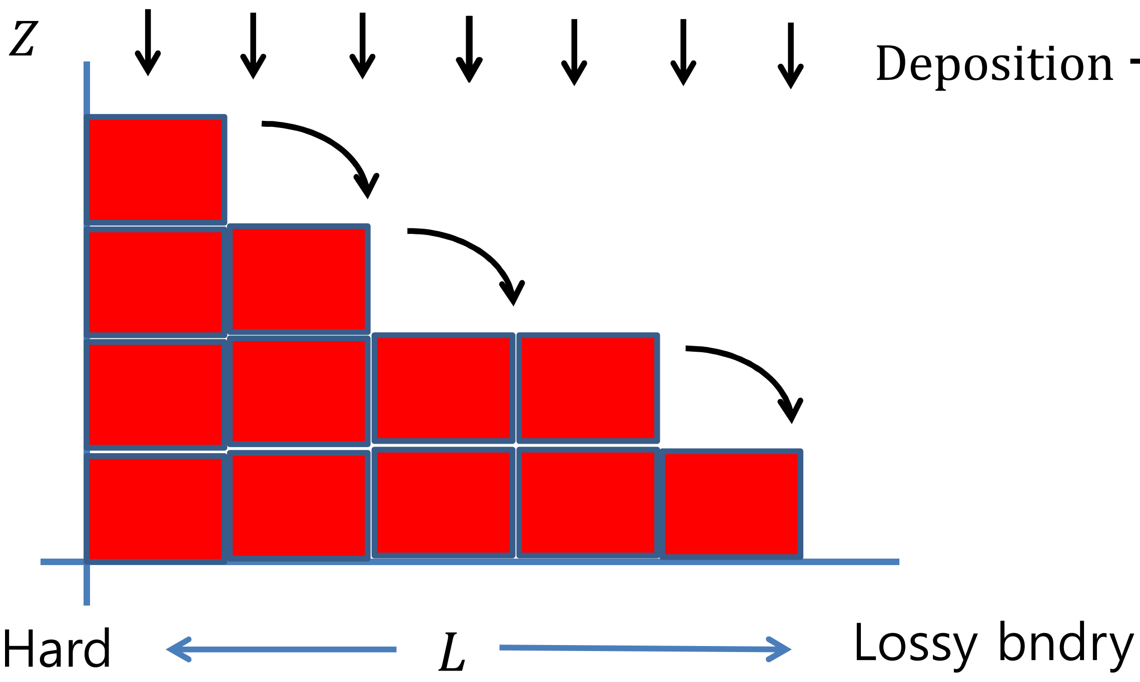


FIG. 1. Self-organized critical state of minimally stable clusters, for a 100×100 array.

- SOC state with minimally stable clusters
- 'Cluster' ≡ set of points reached from toppling of single site (akin percolation)
- Cluster size distribution $D(s) \sim s^{-\alpha}$, $\alpha \sim 0.98$

➔ Zipf, again

- The Classic – Kadanoff et al ’89 1D driven lossy CA



Deposition → random, can profile

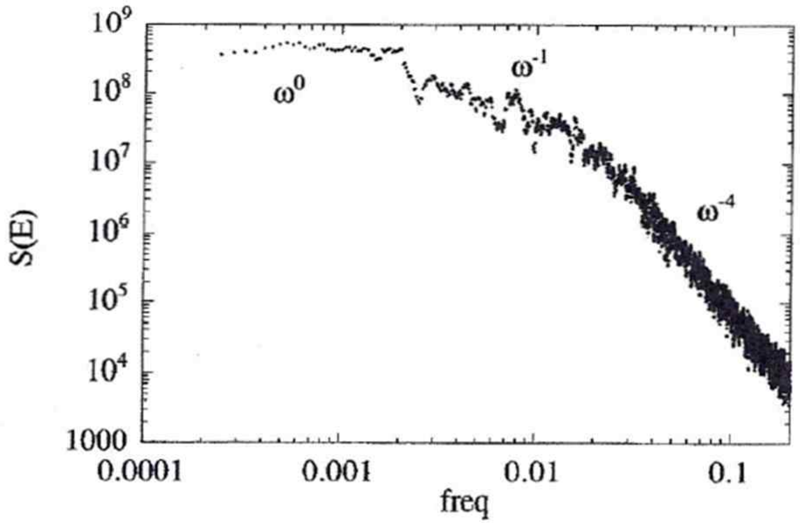
If
$$\begin{cases} Z_i - Z_{i+1} > \Delta Z_{crit} \\ Z_{i+1} \rightarrow Z_{i+1} + N \\ Z_i \rightarrow Z_i - N \\ \text{Etc.} \end{cases}$$

Grains ejected at boundary

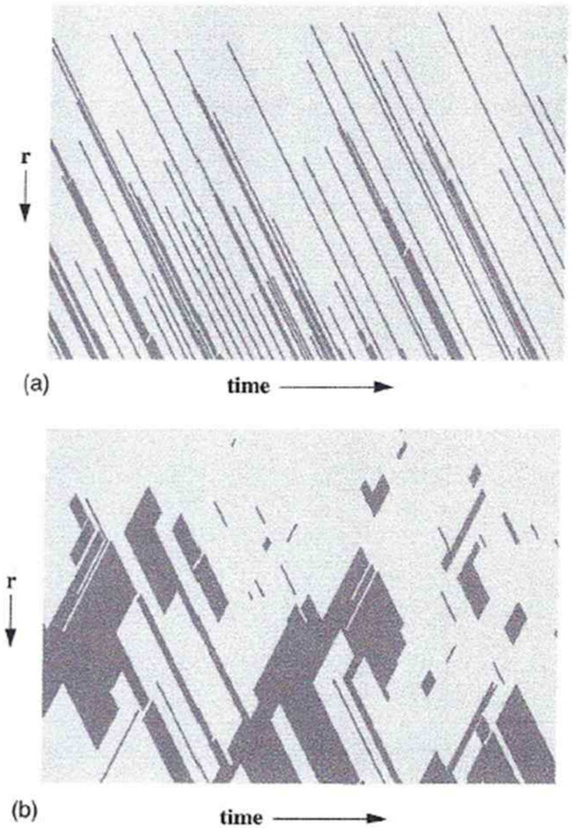
- Interesting dynamics only if $L/\Delta \sim N \gg 1$ ←→ equivalent to $\rho_* \ll 1$ condition – analogy with turbulent transport obvious

TABLE I. Analogies between the sandpile transport model and a turbulent transport model.

Turbulent transport in toroidal plasmas	Sandpile model
Localized fluctuation (eddy)	Grid site (cell)
Local turbulence mechanism:	*Automata rules*:
Critical gradient for local instability	Critical sandpile slope (Z_{crit})
Local eddy-induced transport	Number of grains moved if unstable (N_f)
Total energy/particle content	Total number of grains (total mass)
Heating noise/background fluctuations	Random rain of grains
Energy/particle flux	Sand flux
Mean temperature/density profiles	Average slope of sandpile
Transport event	Avalanche
Sheared electric field	Sheared flow (sheared wind)



Power spectrum of over-turnings $\langle(\Delta Z)^2\rangle_\omega$



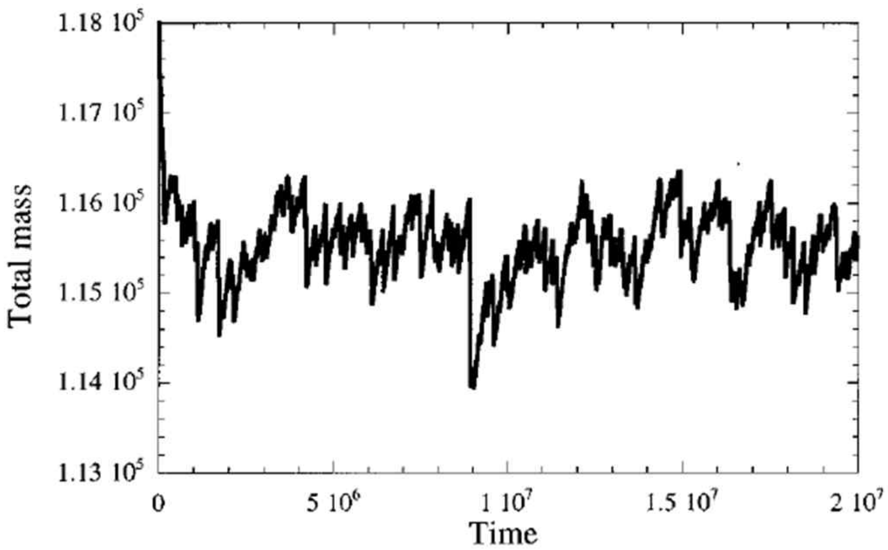
Avalanching

dark → over-turning
light → stable

- Some generic results
  - 1/f range manifest
  - Large power in slowest, lowest frequencies
  - Loosely, 3 ranges:
    - ω^0 → 'Noah'
    - 1/f → self-similar, interaction dominated
    - $1/f^4$ → self correlation dominated
    - Space-time → distribution of avalanche sizes evident

→ Outward, inward avalanching ...

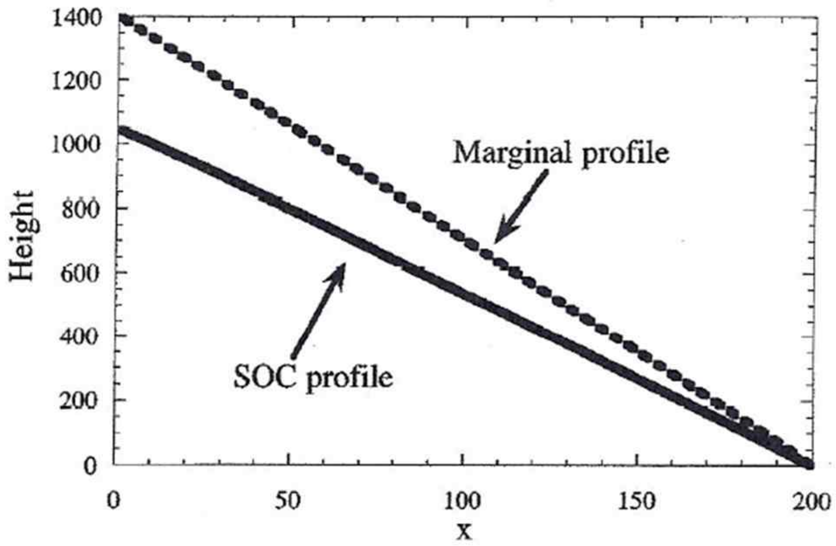
- **Global Structure**



– Time history of total grain content

– Infrequent, large discharge events evident

SOC vs Marginal?



– SOC ≠ Marginal

– SOC → marginal at boundary

– Increasing N_{dep} ➔ SOC exceeds marginal at boundary

– Transport bifurcation if bi-stable rule

– Simple argument for L-H at edge

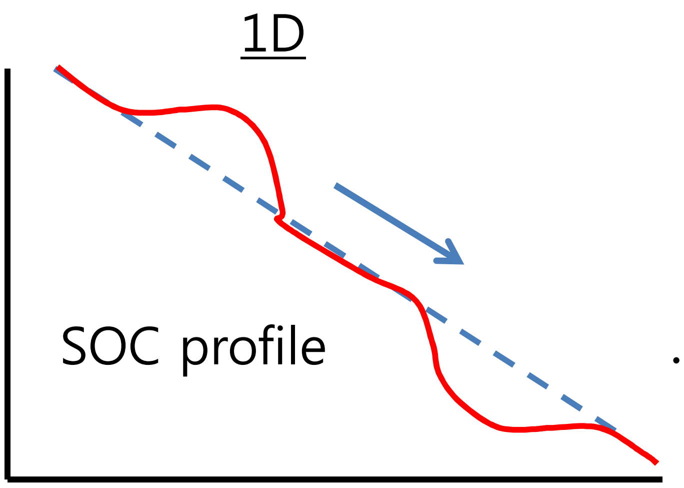
- An Important Connection   Hwa, Kardar '92; P.D., T.S.H. '95; et seq.
  - 'SOC' intimately connected to self-similarity, 'cascade' etc ultimately rooted in fluid turbulence – relate?

And:

  - C in 'SOC' → criticality
  - Textbook paradigm of criticality (tunable) is ferromagnetic ala' Ginzburg, Landau → symmetry principle!?

And:

  - Seek hydro model for MFE connections



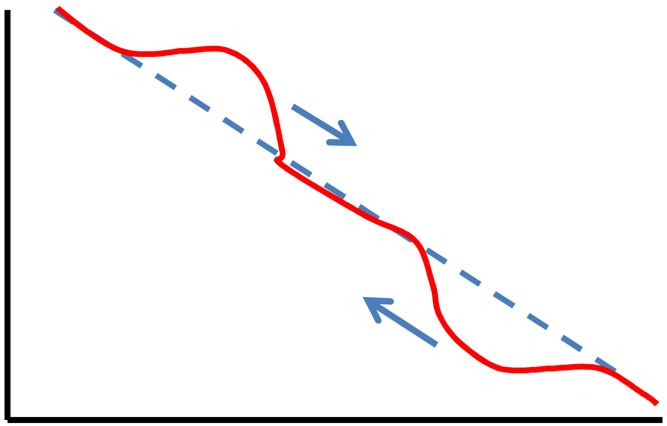
$\delta P \equiv P - P_{SOC}$ → order parameter

→ Local excess, deficit

How does it evolve?

If dynamics conservative;

- $\partial_t \delta P + \partial_x \Gamma(\delta P) - D_0 \partial_x^2 \delta P = \tilde{S}$
- Simple hydro equation
- δP conserved to $\tilde{S}$ boundary
- How constrain δP? → symmetry !
- Higher dimension, $\partial_x \to \partial_\parallel$, and $D_{\perp,0}, \nabla_\perp^2$ enter



$\delta P > 0$ → bump, excess

→ Tends move down gradient, to right

$\delta P < 0$ → void, deficit

→ Tends move up gradient, to left

- Joint reflection symmetry principle

$x \rightarrow -x$

$\delta P \rightarrow -\delta P$

➔ $\Gamma(\delta P)$ unchanged

i.e. flip pile, blob

→ void structure → rt.

- Allows significant simplification of general form of flux:

$$\Gamma(\delta P) = \sum_{m,n,q,r,\alpha} \{A_n(\delta P)^n + B_m(\partial_x \delta P)^m + D_\alpha(\partial_x^2 \delta P)^\alpha + C_{q,r}(\delta P)^q(\partial_x P)^r + \cdots\}$$

- So, lowest order, smoothest model:

  $\Gamma(\delta P) \approx \alpha\, \delta P^2 - D\partial_x \delta P;\quad \alpha, D$ coeffs as in G.-L.

  N.B.: Heuristic correspondence

$$\alpha\delta P^2 \leftrightarrow -\chi\left(\frac{1}{P}\nabla P\Big|_{threshold} - \frac{1}{L_{P_{crit}}}\right)\nabla P$$

  And have:

$$\partial_t \delta P + \partial_x(\alpha\delta P^2 - D\partial_x \delta P) = \tilde{s}$$

  - Noisy Burgers equation
  - Solution absent noise → shock
  - Shock ←→ Avalanche
  - Manifests shock turbulence → widely studied

- More on Burgers/hydro model (mesoscale)
  - Akin threshold scattering
  - $V \sim \alpha\, \delta P$ relation → bigger perturbations, faster, over-take
  - Extendable to higher dimensions
  - Cannot predict SOC state, only describe dynamics about it. And α, D to be specified
  - $\langle \delta P \rangle$? → corrugation (!?)
  - Introducing delay time → traffic jams, flood waves, etc (c.f. Whitham; Kosuga et al '12)

- Avalanche Turbulence
  - Statistical understanding of nonlinear dynamics → renormalization
  - Conserved order parameter

$$\partial_x (\alpha \delta P^2) \rightarrow \nu_T k^2 \, \delta P_k$$

$$\nu_T \approx \left(\alpha^2 S_0^2 \int_{k_{min}}^{0} dk / k^4 \right)^{1/3} \rightarrow (\alpha^2 S_0^2)^{1/3} k_{min}^{-1}$$

$$\sim (\alpha^2 S_0^2)(\delta l)$$

Infrared divergence due slow relaxation

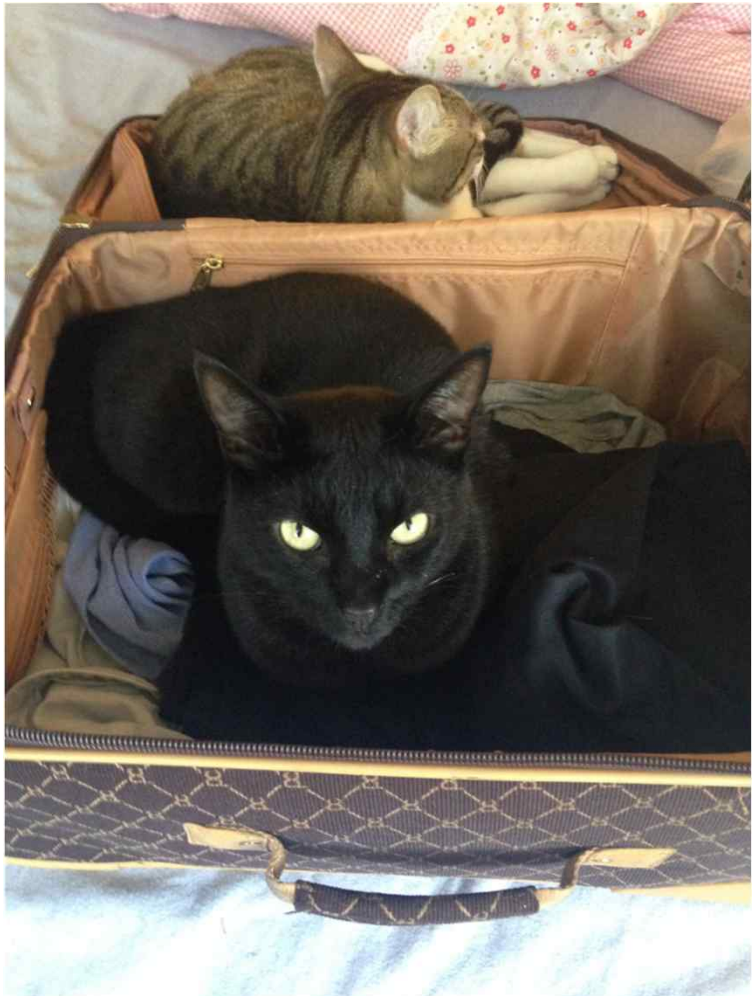
  - $(\delta l)^2 \sim \nu_T \delta t$ → $\delta l \sim \delta t$
    - H → 1
    - 'Ballistic' scaling

- Infrared trends ←→ non-diffusive scaling, recover self-similarity
- Amenable to more general analyses using scaling, RG theory
- Pivotal element of 'SOC' theory as connects 'SOC' world to turbulence world, and enables continuum analysis

‘SOC’ in MFE – A Selective OV

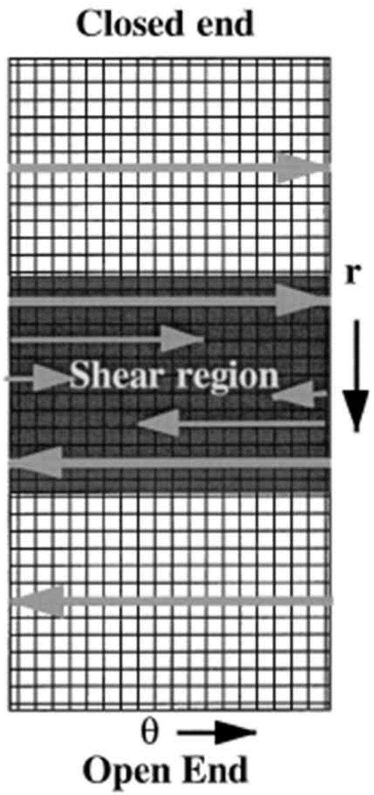
- New model → sheared sandpile ➔ illustrates physics of avalanches
- Going beyond the Box
  - Simulations continuum, flux-driven

    → avalanches really happen!
- Some findings from fluctuations:
  - Hunting for evidence of H in L



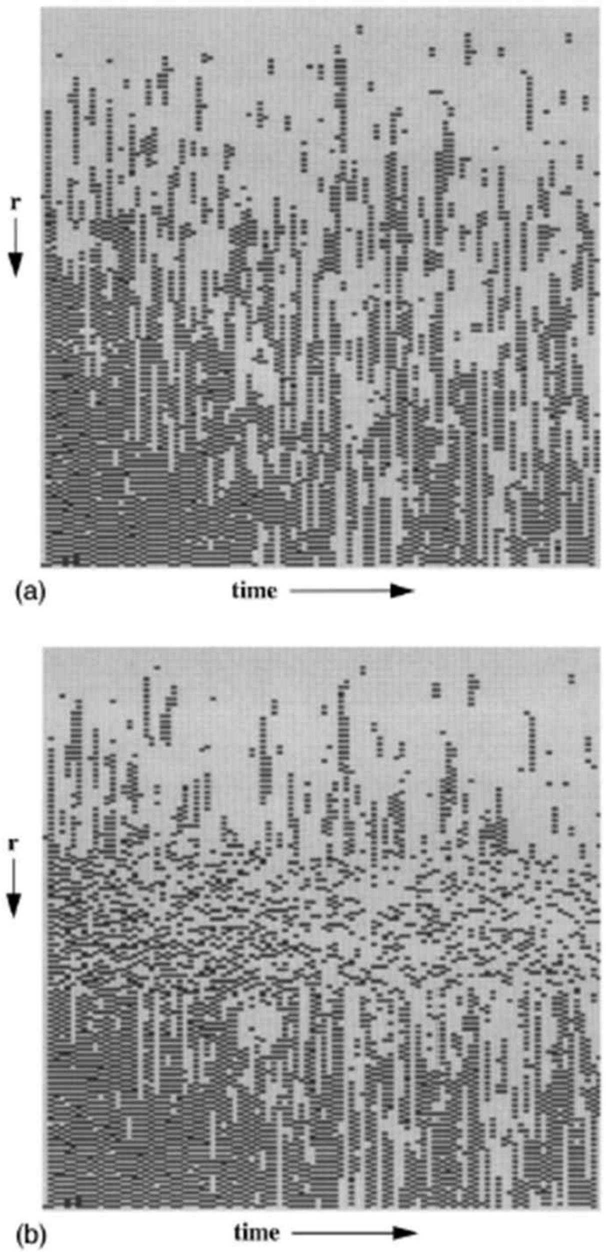
(Newman, BAC, P.D., T.S.H. '96)

Sheared Sandpile



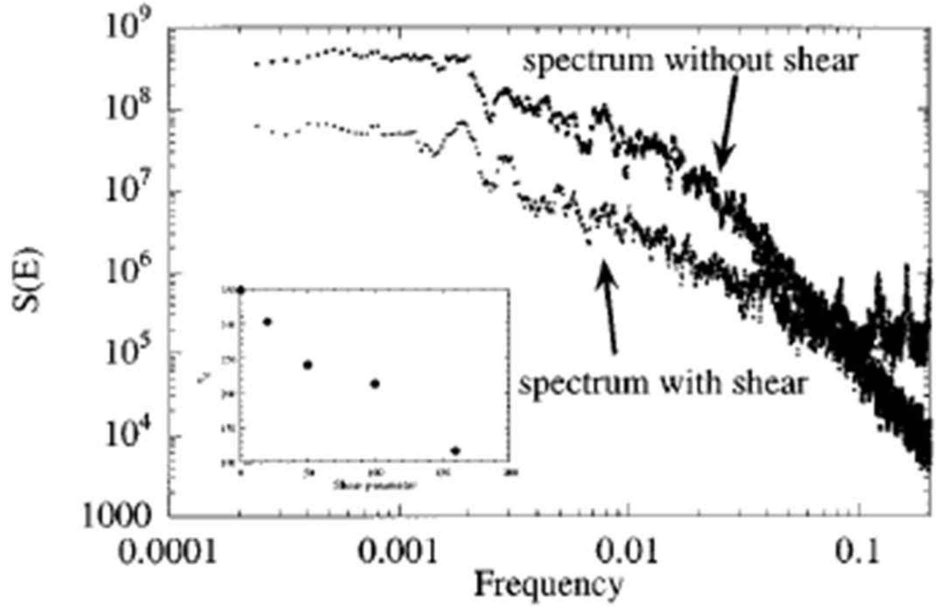
- Shear imposed in finite region
- $\omega_s \sim V_E' \Delta_s$

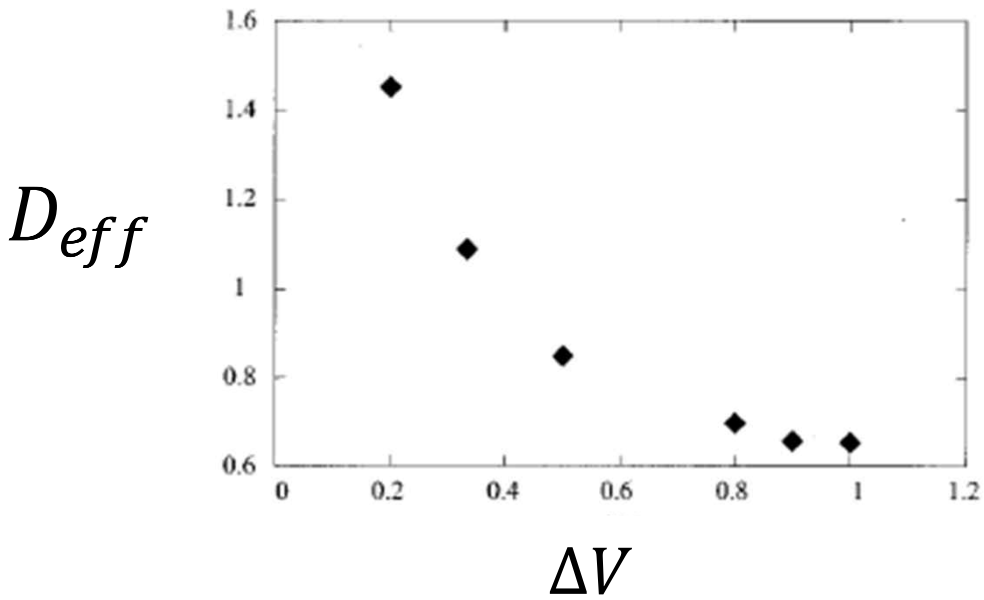
What Happens?



- Shear imposed in finite region
- $\omega_s \sim V_E' \Delta_s$
- Shading as before
- Illustrates every 50th step
- Overturning persists (dark sites) in shearing zone, but: avalanche coherence broken

$\langle \tilde{N}^2 \rangle_\omega$ spectra (Rules fixed)





- Low frequency content <u>drops</u>

But

- High frequency content <u>increases</u>

←→Persistence of overturning

➔ Extended discharge events suppressed

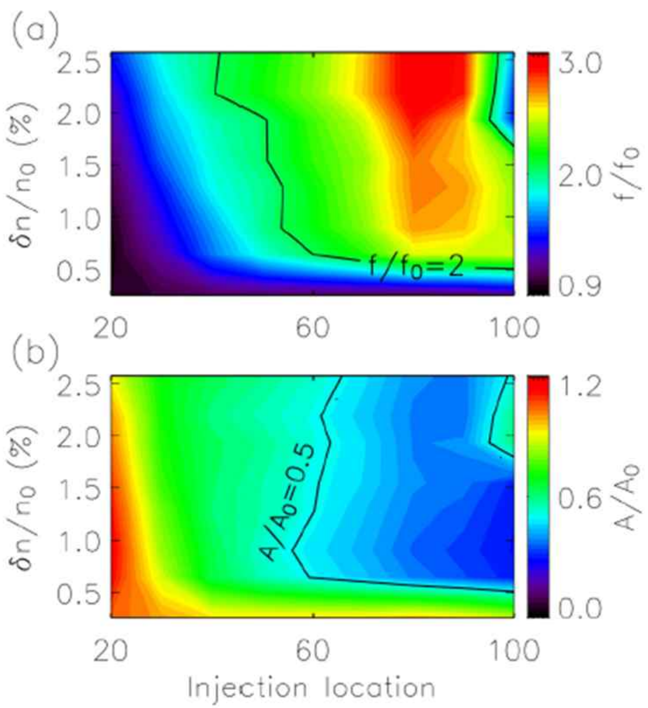
- Can map results to drop in D_{eff}

More in this vein

I. Gruzinova, P.D. (2002)

- Bi-stable pile ←→ toppling rules
- "H-mode" barrier triggered naturally, builds inward
- Adding ambient D_0 and ∇N limit covers 'ELM cycle' etc.

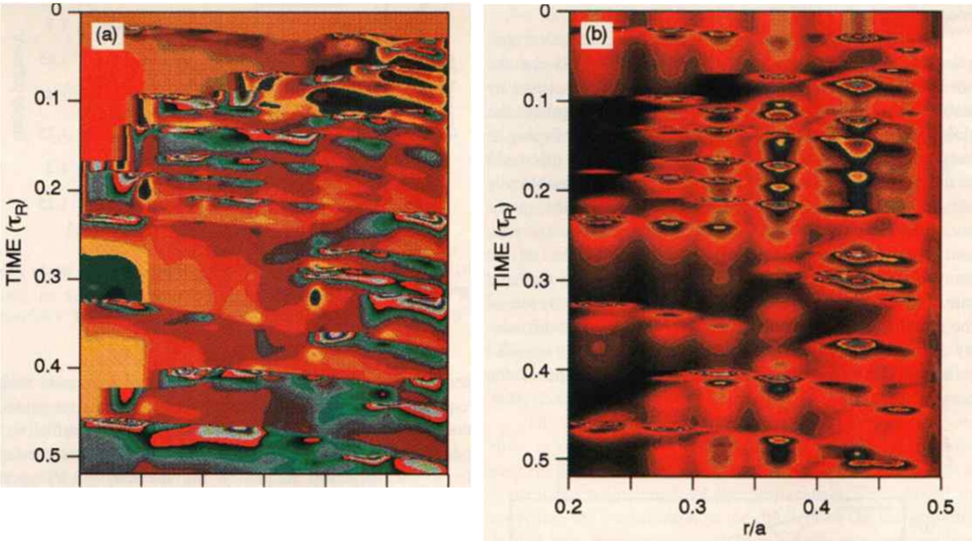
T. Rhee, J.M. Kwon, P.D., W. Xiao (2012)



- Extend above genre model to further explore 'ELM' discharges
- Demonstrated grain injection ala' SMBI can break up avalanches
- Identified 'sweet spot'

- "Why don't you guys think outside the (sand) box and do real science?"
- Simulations! (continuum)
  - (BAC, et al '96) Flux driven resistive interchange turbulence; "weak drive"
    - Noisy source: $S_0 = S(r) + \tilde{S}$ (noise)
    - Reynolds stress driven flows, viscosity
    - Threshold: ala' Reyleigh, ∇P vs ν_u, D_c
  - Flux drive, fast gradient evolution essential, as $V_{aval} \leq V_*$

Some Findings: Avalanches happen!



– Clear difference in <u>upper</u> pressure contours

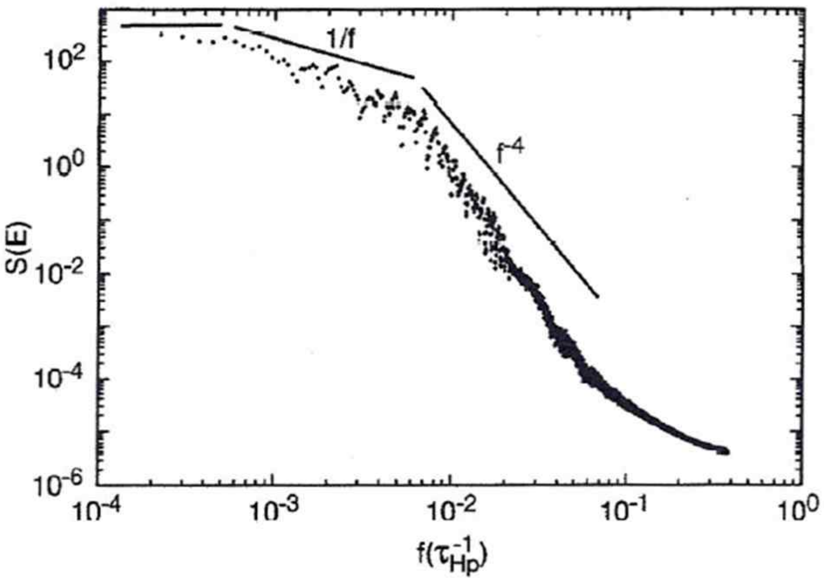
vs <u>lower</u> $(e\phi\,/T)_{rm\,s}$ contour

– Avalanches evident in δP

But

– Modes, resonant surfaces in $e\phi\,/T$

➔ illustrates collective character of avalanches

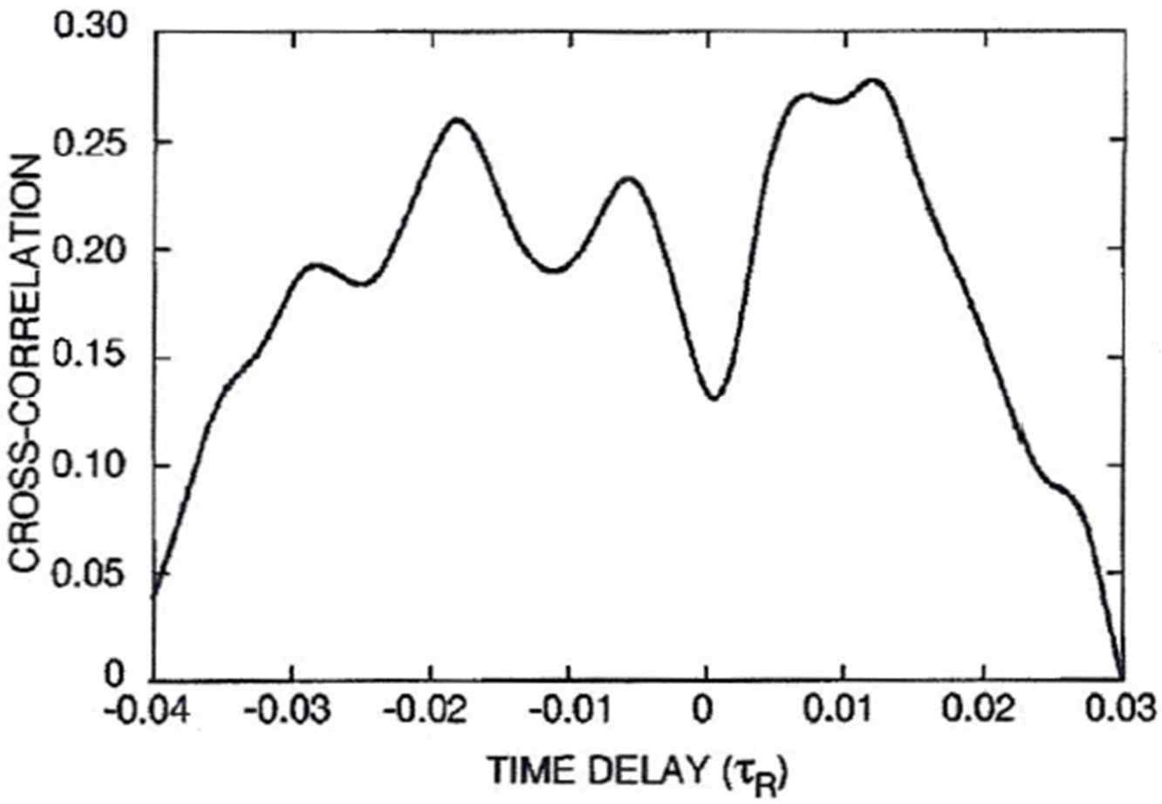


– 1/f recovered in $\langle\left(\frac{e\tilde{\phi}}{T}\right)^2\rangle_\omega$

– Very similar to pile

– Later observed in flux

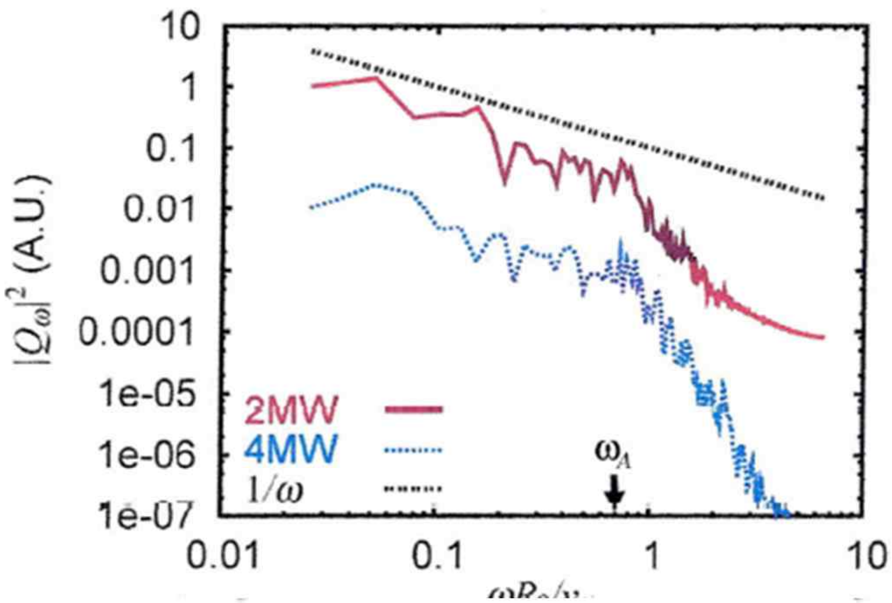
- 2 peaks in cross correlation of low frequency modulation



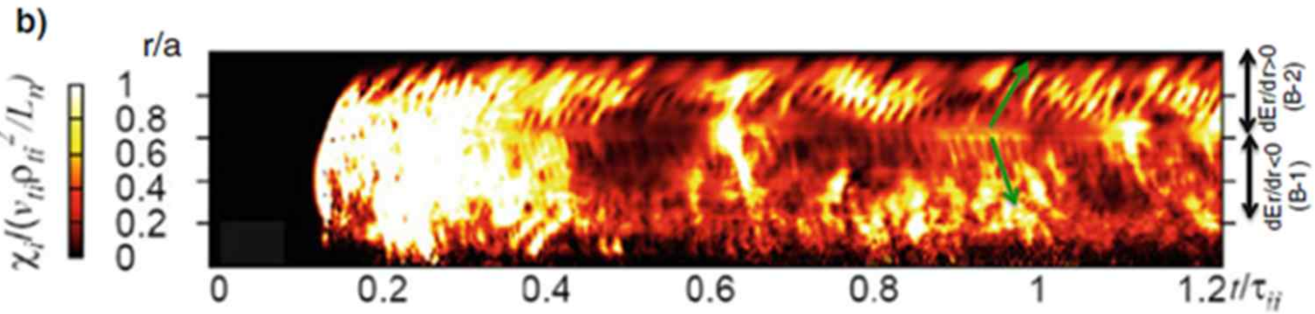
2 peaks → ingoing, outgoing avalanches

- Shear flow can truncate avalanches, ala' pile

But <u>real</u> men do gyrokinetics !



- Idomura, et al (2009)
  - Flux driven ITG, GT5D
  - Also explored intrinsic flow
- 1/f evident in $|Q_\omega|^2$

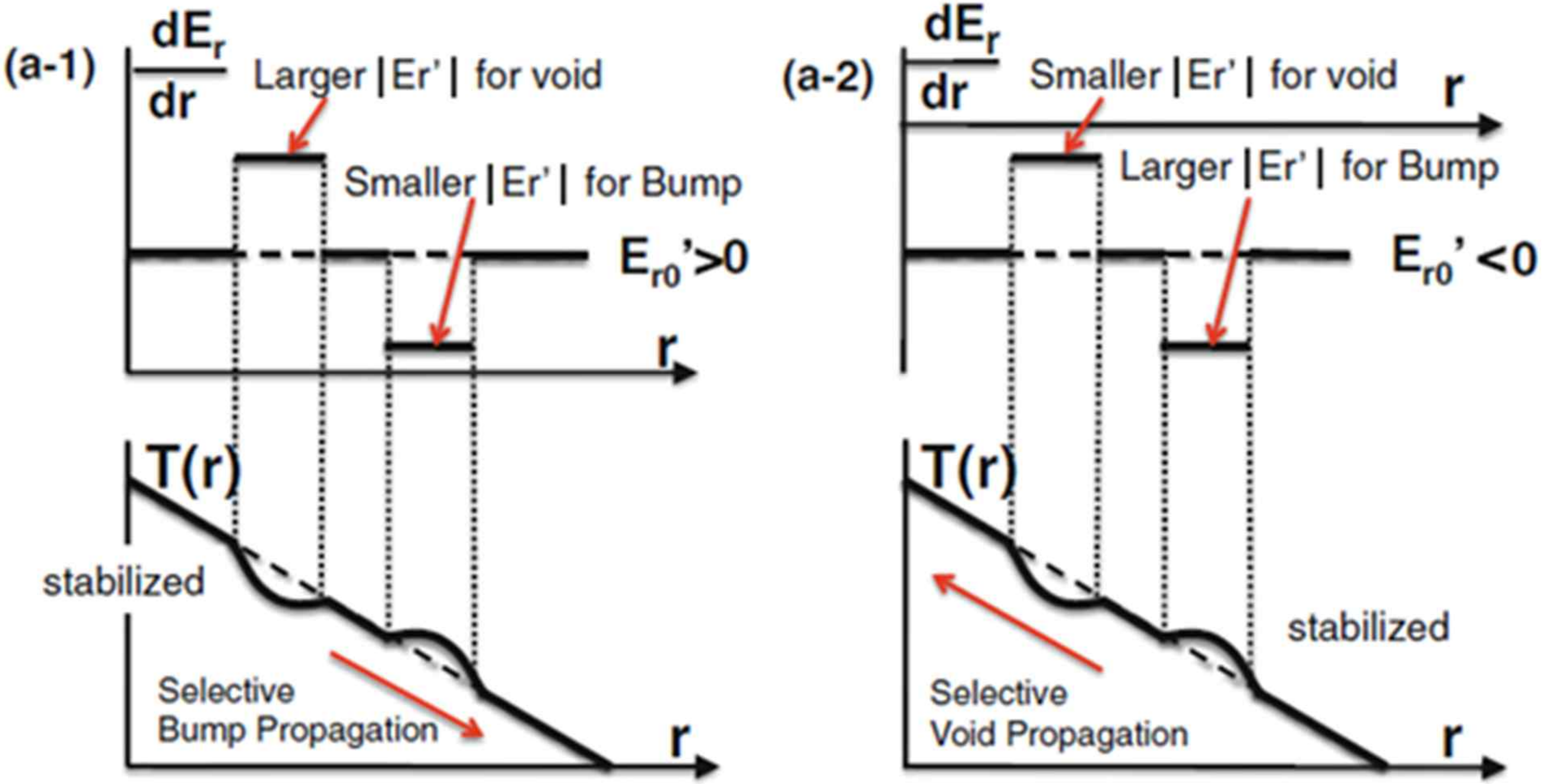


- $f^0, 1/f, 1/f^\alpha$ $(\alpha \gg 1)$ ranges, ala' pile and g-mode.

  Sic transit gloria GK

- If SOC profile ≈ Marginal profile

  can link E_r' to bump/void imbalance (Idomura, Kikuchi)

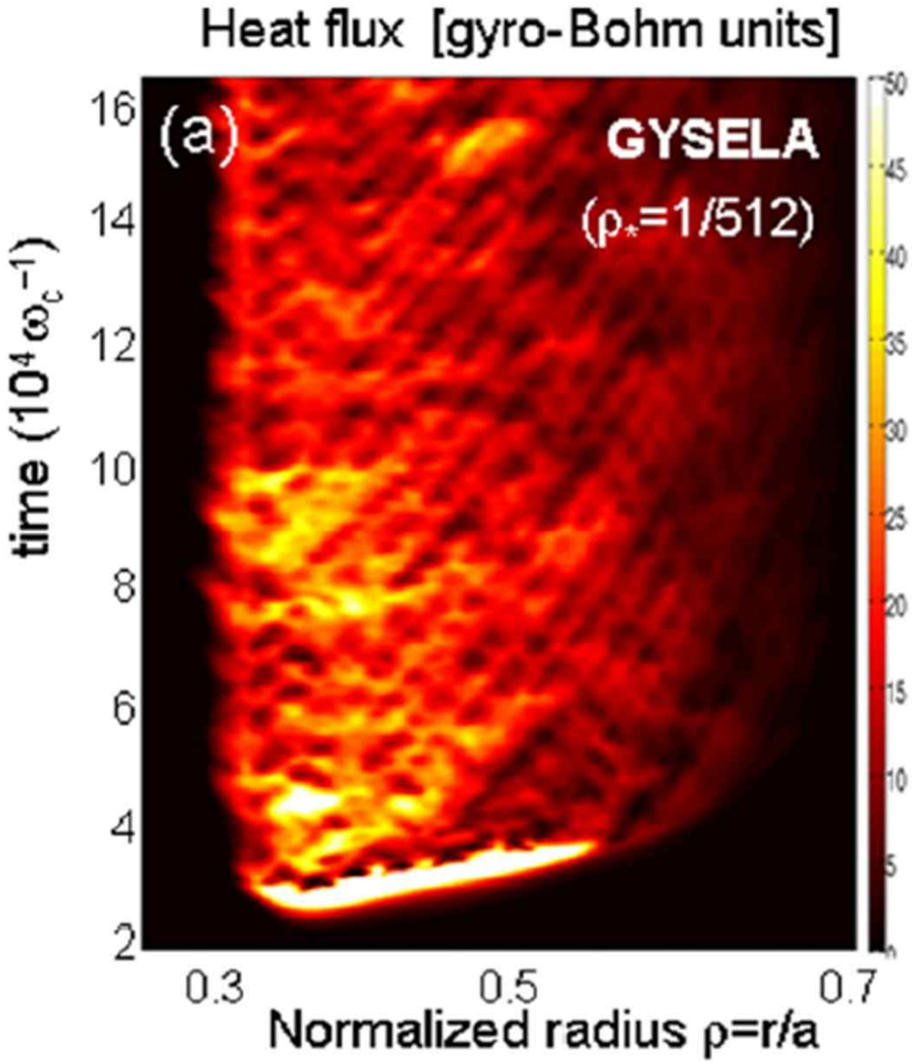


➔ Blobs dominate, $E_r' > 0$

Voids dominate, $E_r' < 0$

- GYSELA Results: Avalanches Do ‘matter’

GYSELA, rhostar=1/512 [Sarazin et al., NF 51 (2011) 103023]



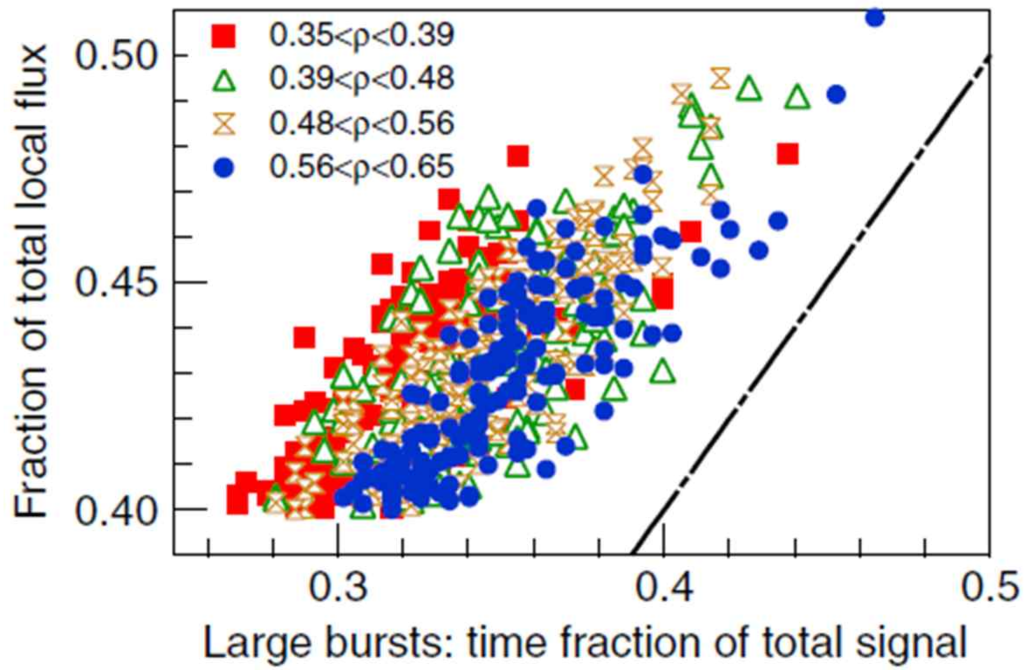


Figure 2. Fraction of the local radial turbulent heat flux carried out by a certain fraction of the largest scale bursts, as estimated from figure 1(a) (GYSELA data). Each point refers to one specific radial location. The colours allow one to distinguish four different radial domains. The considered time series ranges from $\omega_{c0}t = 56\,000$ to $\omega_{c0}t = 163\,000$.

- Distribution of Flux Excursion and Shear Variation

GYSELA, rhostar=1/64 [Sarazin et al., NF 50 (2010) 054004]

Asymmetry!

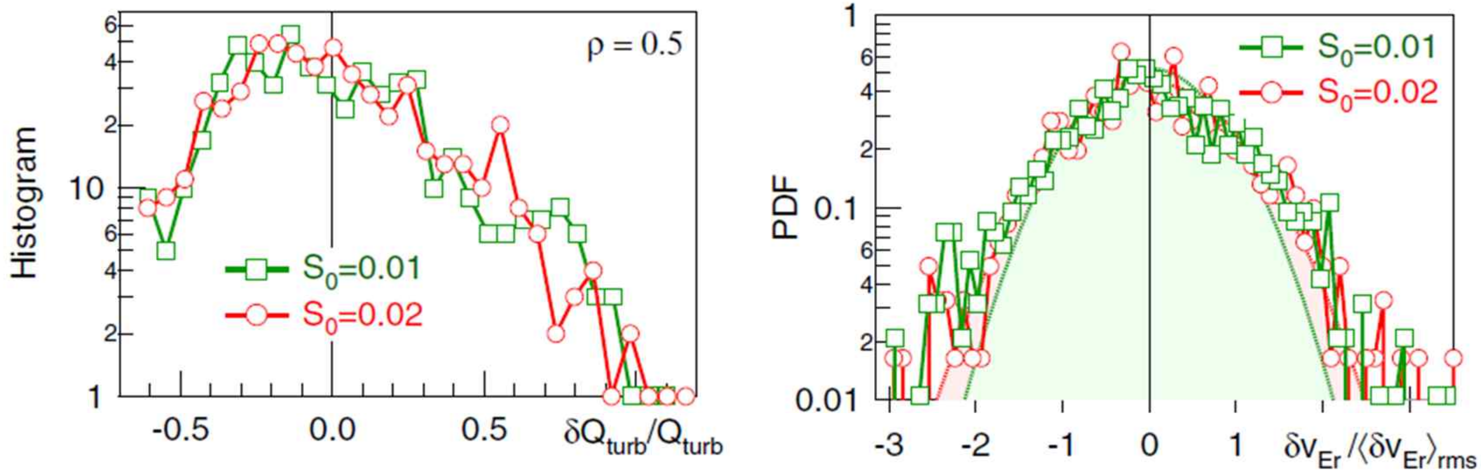


Figure 7. (Left) histogram of the turbulent heat flux $Q_{\rm turb}$ at $\rho = 0.5$ for two magnitudes of the source ($\rho_* = 1/64$). $\delta Q_{\rm turb}$ stands for the difference between $Q_{\rm turb}$ and its time average, taken over the entire non-linear saturation phase. (Right) corresponding PDF of the fluctuations of the radial component of the electric drift. (Colour online.)

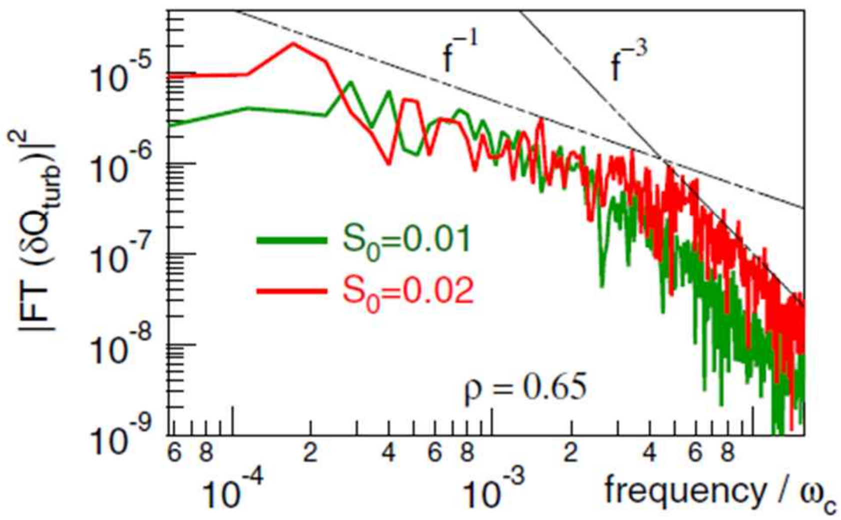


Figure 8. Frequency Fourier spectrum of the turbulent heat flux at $\rho = 0.65$ for two magnitudes of the source ($\rho_* = 1/64$). (Colour

- Experiments !?
  - Several studies of H-exponent for edge turbulence in 'boring plasmas' ➔ Van Milligan, Carreras, Hidalgo et al
  - Data via Langmuir probes …
  - Non-trivial analysis …

cf: Direct imaging of avalanches beyond current capabilities

- Obvious need for more here, integrated into overall confinement picture
- Ideal topic for HL-2A, J-TEXT, PANTA

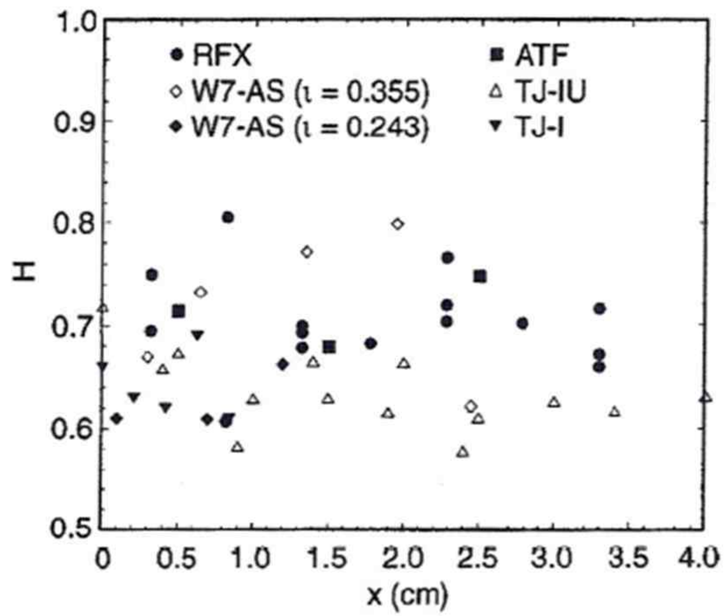
- Tabulated results (Carreras et al '98)

  - $H \approx 0.7$ is a general trend

TABLE III. A summary of the analysis results.

Device	Number of time series	$\langle H\rangle_{in}$	$\langle H\rangle_{out}$	τ_D (μs)	Self-similarity range (ms)
TJ-I	9	0.64±0.03	0.70±0.04	3.0	0.02–1.0
JET limiter	4	···	0.52±0.04	29.0	0.1–2.0
JET divertor	4	···	0.63±0.03	19.0	0.1–2.0
TJ-IU	21	0.64±0.03	0.67±0.01	6.0	0.1–2.0
W7-AS ι_a=0.243	24	0.62±0.01	0.60±0.04	20.0	1–20
W7-AS ι_a=0.355	29	0.72±0.07	0.66±0.06	19.0	1–20
ATF	20	0.71±0.03	0.92±0.07	34.0	1–12
RFX	29	0.69±0.04	···	3	0.03–3.0
Thorello	10	0.55±0.04	···	6	0.05–5.0

  - Range of H values from separatrix -> in
  - $H \approx 0.7$
  - significant that $H > 0.5$, always



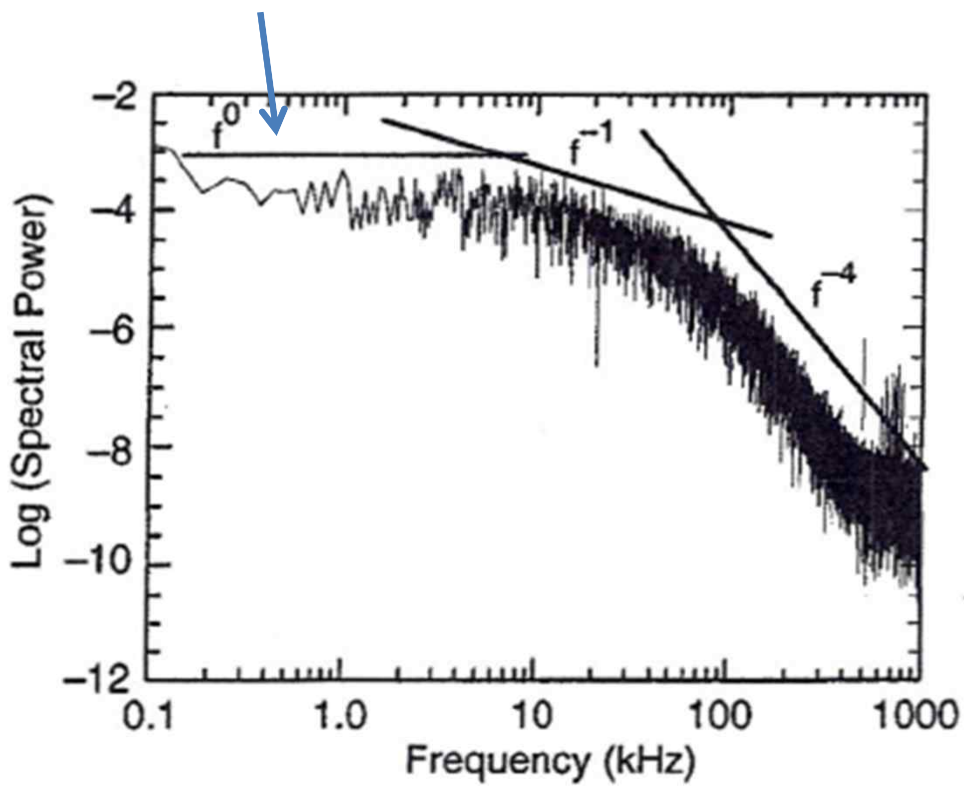
– $\langle \tilde{I}_{sat}^{2} \rangle_{\omega}$ spectrum from W7-AS

– Familiar spectral structure,

pile – fluid – GK – reality

– Universality?!

'Noah'!



- Little offered on:
  - Flow shear effects on H
  - Correlation of trends in confinement with trends in H

Advertisement

- Related subject: ‘nonlocality’ phenomena
  - See OV article by K. Ida et al NF 2015
  - Relation of SOC and ‘SOC’ to nonlocality → Discussion

‘SOC’ and Kinetics: A Very Brief introduction to CTRW and FK

- CTRW ≡ continuous time random walk (Klafter, Montroll)

  FK ≡ Fractional Kinetics (Zaslavsky)

- Conceptually straightforward, but highly technical subject

  For short intro: see week 8 notes by Kurt Thompson, P.D., UCSD Physics 235, Spring 2016

- Message: Not a panacea

- Recall: Fokker-Planck Theory

dist. 'back' 1 step

$$f(t+\Delta t) = \int d(\Delta x)\mathrm{T}(x, \Delta x, \Delta t) f(x - \Delta x, t)$$

transition probability (input)

Expansion

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{\langle \Delta x \rangle}{\Delta \mathrm{t}} f - \frac{\partial}{\partial x}\left(\frac{\langle \Delta x \Delta x \rangle}{2\Delta t} f\right)\right\}$$

$\frac{\langle \Delta x \rangle}{\Delta \mathrm{t}} \to V$, $\frac{\langle \Delta x \Delta x \rangle}{2\Delta t} \to D$

- Key elements:
  - Existence of variance of T? → i.e. is $\langle \Delta x^2 \rangle$ finite?
  - Is Δt regular or irregular?

- The point:
  - For boring Gaussians, variance converges
  - For 'SOC' → self-similarity → power laws →

$$\int dx \, \frac{x^2}{x^{1+\alpha} + \cdots}$$ → TROUBLE, unless $\alpha > 2$

  - Welcome to the 'Fat Tails' problem!

"Life always has a fat tail."

Eugene Fama; Nobel in Economics, 2013

  - Enter the Levy Flight (Random walk with infinite variance)

- **Pareto-Levy Distributions**

  - Gaussian is only 1 of, and only, case with finite variance, of infinite number of stable distributions

  - Easier to work with generating function:

    $P_\alpha(q) = \exp[-c|q|^\alpha]$, Levy distribution, index α

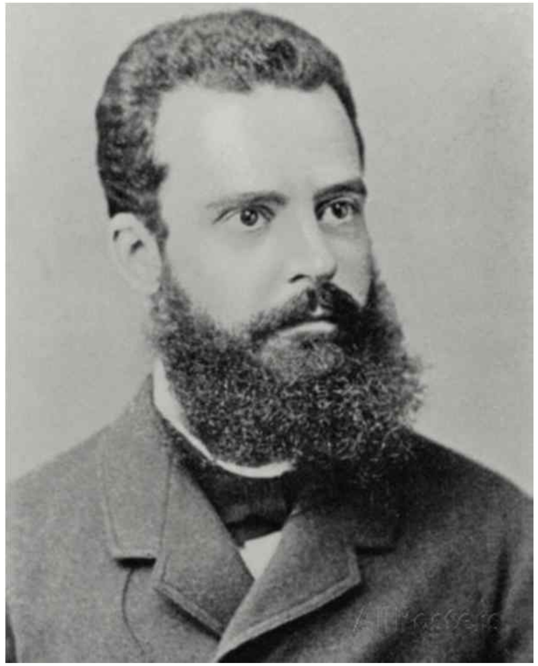
    $\alpha = 2$ → Gaussian

    and

    $P(q,t) = \exp[-c\,t\,|q|^\alpha]$

    $\alpha = 2, C \to D$ → diffusion G.F.

    $P_\alpha(x,t) \underset{x \to \infty}{\sim} t/|x|^{\alpha+1}$ → "accelerating tail"



Vilfredo Pareto
→ 1897:
Power law tail
In wealth distribution
$1 < \alpha < 2$

- Can you give us some physical insight into 'flight'?

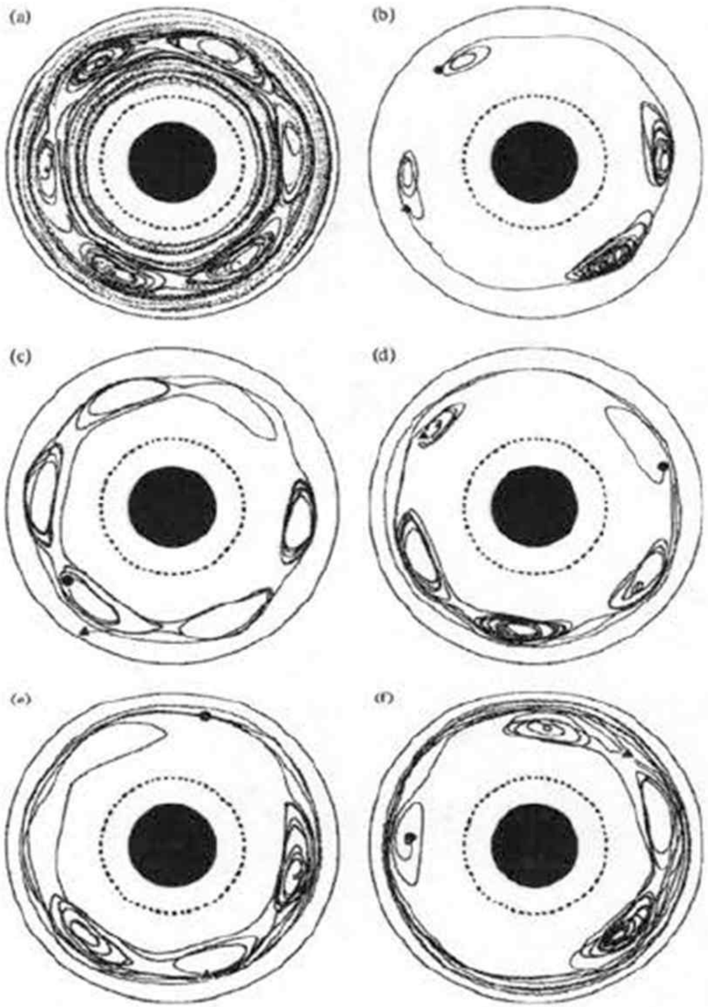
Observation of Anomalous Diffusion and Lévy Flights in a Two-Dimensional Rotating Flow

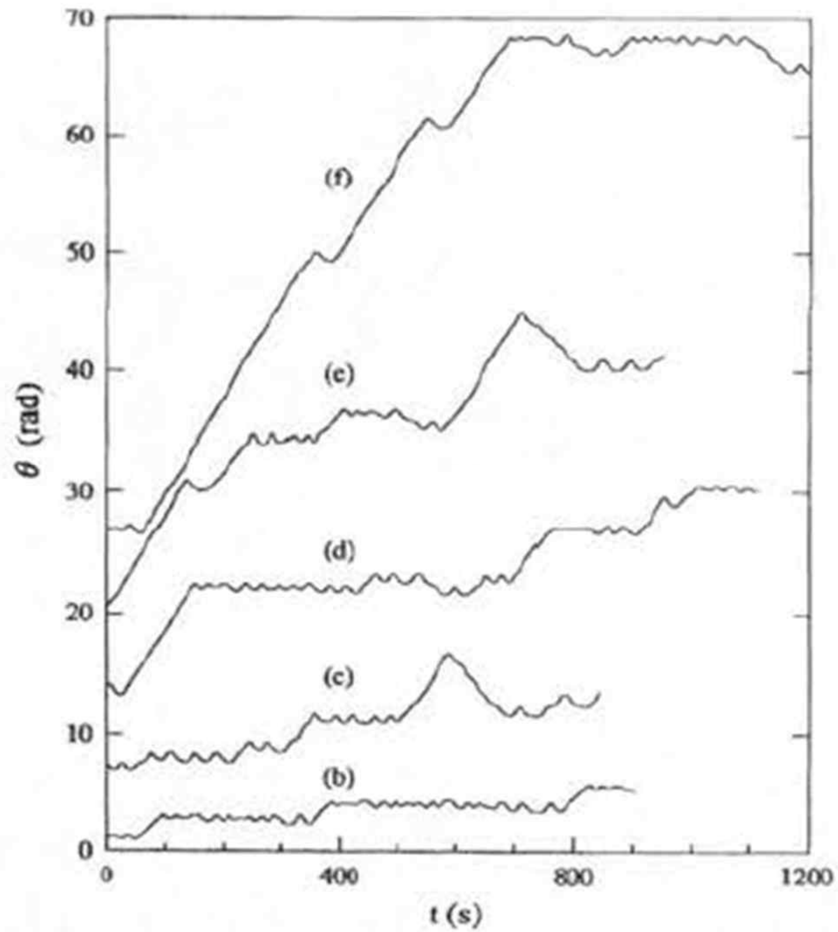
T. H. Solomon,* Eric R. Weeks, and Harry L. Swinney†
Center for Nonlinear Dynamics and Department of Physics, University of Texas at Austin, Austin, Texas 78712
(Received 17 September 1993)

Chaotic transport in a laminar fluid flow in a rotating annulus is studied experimentally by tracking large numbers of tracer particles for long times. Sticking and unsticking of particles to remnants of invariant surfaces (Cantori) around vortices results in superdiffusion: The variance of the displacement grows with time as t^γ with $\gamma = 1.65 \pm 0.15$. Sticking and flight time probability distribution functions exhibit power-law decays with exponents 1.6 ± 0.3 and 2.3 ± 0.2, respectively. The exponents are consistent with theoretical predictions relating Lévy flights and anomalous diffusion.

- Traced particle dynamics in rotating flow, with vortex array

- Upshot is strongly non-diffusive behavior





- $\langle \delta x^2 \rangle \sim t^{1.6}$
- Evident that anomalous exponent due to prolonged sticking, with occasional long steps (flights)
- PDF suitably distorted

- CTRW and FK
  - Aim to extend Fokker-Planck approach to Levy Distributions
- Approaches:
  - CTRW: distribute Δt, with fat tail

    i.e. $T(x, \Delta x) \rightarrow T(x, \Delta x, t, \Delta t)$

    allows prolonged sticking times
  - FK: treat $\Delta t, \Delta x$ as powers

    i.e. $\Delta x \frac{\partial f}{\partial x} \rightarrow (\Delta x)^{\alpha} \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$, etc

    accommodates rough(fractal) distributions

- Very over-simplified bottom line:

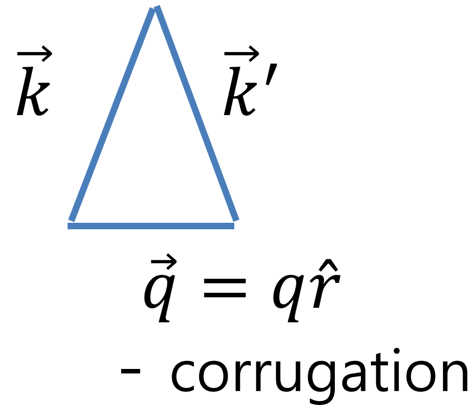
Parameter	Fokker-Planck	Fractional Kinetics
Stochastic variable	Δx	Δx, Δt
Role of time	Fixed clock	Variable, PDF
Variance	$\langle \lvert x \rvert^2 \rangle \sim t$	$\langle \lvert x \rvert^2 \rangle \sim t^\mu$ where $\mu < 2$
Kolmogorov Conditions	Equation (12)	Equation (21)
$A(y, \Delta t)$	$\langle\langle(\Delta y)\rangle\rangle$	No simple form
$B(y, \Delta t)$	$\langle\langle(\Delta y)^2\rangle\rangle$	$\frac{\langle\langle \lvert \Delta x \rvert^{\alpha_1} \rangle\rangle}{\Gamma(1+\alpha_1)}$
Relation between $\mathcal{A}(x)$ and $\mathcal{B}(x)$	Equation (10)	Equation (22)

- Needed input:
  - μ → set by critical exponents for space time
  - A, B → scalings set by pdf
- Underlying physical model sets outcome

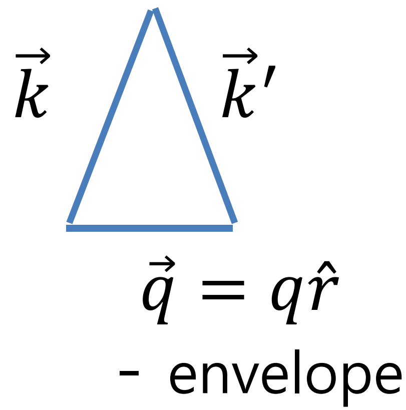
- **FAQ's re: SOC and 'SOC'**
  - Are tokamak profiles a SOC?
  - Is 'SOC' useful?
  - What have we learned from 'SOC' studies?
  - How are avalanches related to t.s.?
  - What can we predict with 'SOC'?
  - How does 'SOC' help analysis and modelling?
  - Relation to 'SOC' to 'Non-Local Transport'?

• Turbulence Spreading vs Avalanching

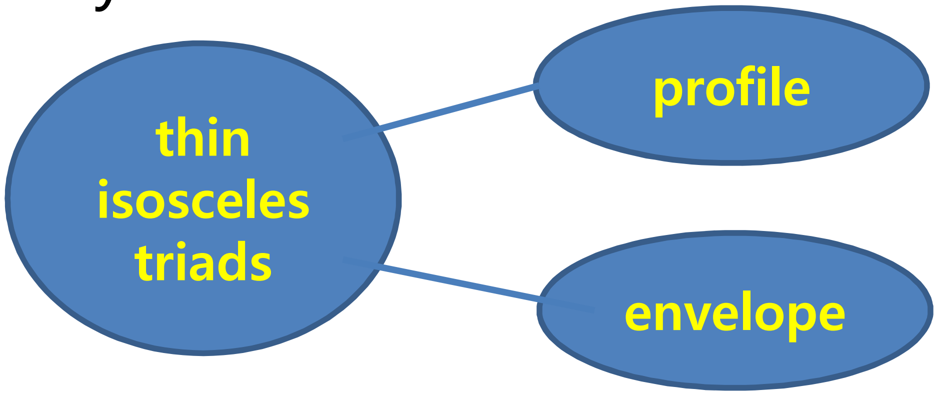
- Both: (non-Brownian) radial propagation of excitation
- Avalanching:
  - via overturning and mixing of neighboring cells
  - Coupling via $\nabla\langle P\rangle$
  - $\partial_t \delta P \sim \partial_x \, (\alpha \delta P^2)$



- Turbulence spreading (t.s. by T.S.)
  - via spatial scattering due nonlinear coupling
  - Couple via turbulence intensity field
  - Usually $\partial_t I \sim \partial_x (D_0 I \partial_x I)$



- **Bottom Line:**
  - Very closely linked



  - ~ impossible to have one without other
  - t.s. can persist in strong driven, non-marginal regimes
  - Which effect more dramatic is variable → specifics?
  - Controversy sociological (or sociopathic)…

- **Suggestions**
  - Can generate a LONG list…
  - Some standouts:
    - Integrate H-exponent studies, etc. with over-all picture of confinement trends. Special focus ←→ flow shear
    - Elucidate systematics of SOC profile vs marginal profile. What, really, is stiff?
    - Predict avalanche outer scale → staircase

- **Suggestions**
  - Re FK:
    - Physical insight into distribution, critical exponents
    - Simple model in spirit of Dupree '66
  - Fate of avalanches, etc. in multi-scale or ITG-TEM systems. Treat ions as full-f, flux driven, electrons as δf ?!
  - How does H behave approaching I, H?

References:

- Many books, reviews; see especially:

“Self-Organized Criticality”

- H.J. Jensen, (CUP)

- See also:

http://physics.ucsd.edu/students/courses/spring2016/physics235/

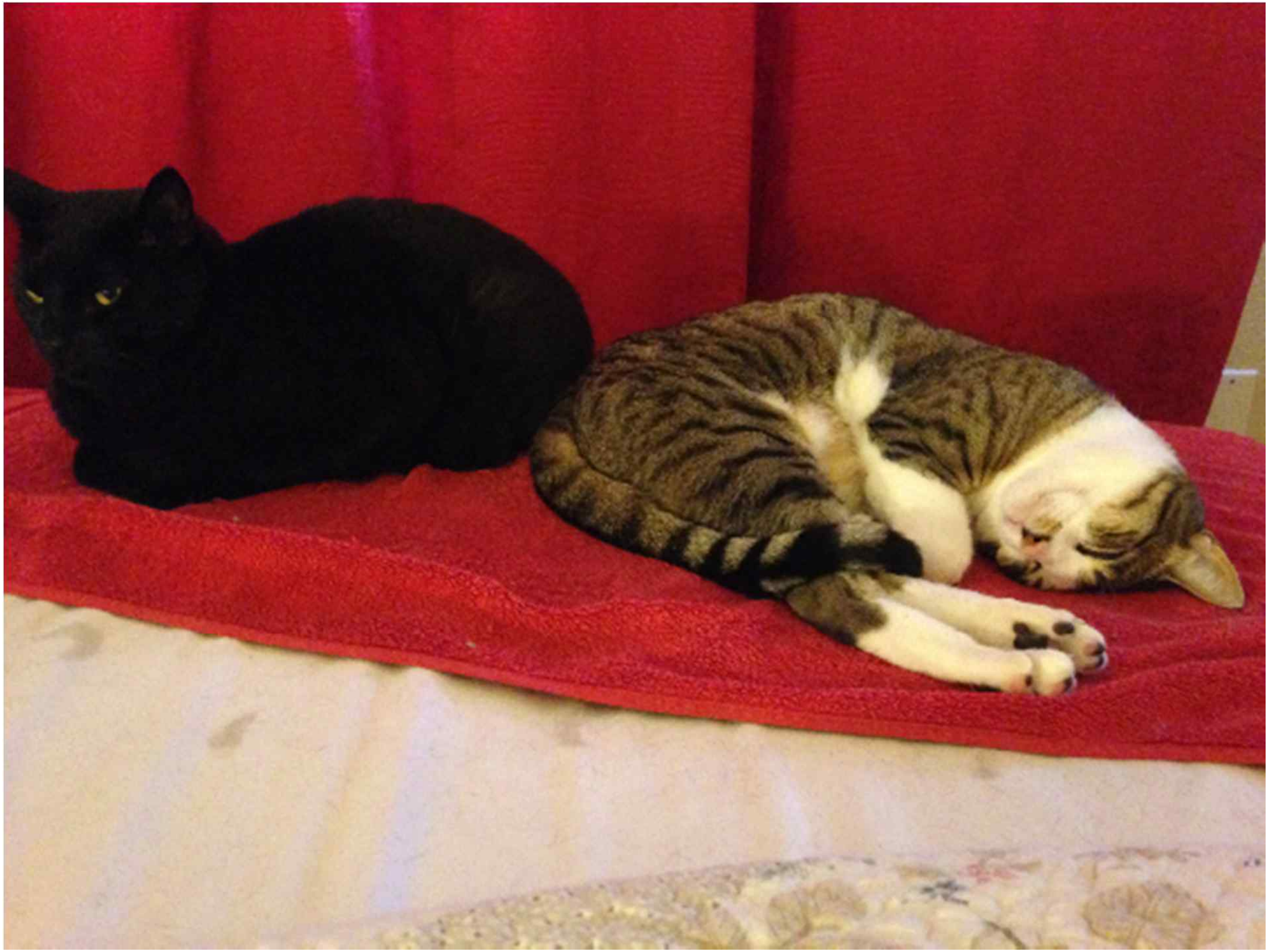
for class materials: notes, summaries, and extensive reprint collection.

- **Concluding Thoughts**
  - SOC and 'SOC' have been fun to work on, for 21+ years.
  - Thanks to:

  → Collaborators, including: T.S. Hahm, B.A. Carreras, O.D. Gurcan, J.M. Kwon, T. Rhee, Y. Kosuga, W. Xiao, Y. Xu, C. Hidalgo

  → Physics 235 Students, UCSD, spring 2016

  - N.B.: 2017 will mark 30 years since BTW → intresting conferences ahead…



- This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Science, under Award Number DE-FG02-04ER54738